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LETTER TO THE EDITOR

A Monte Carlo renormalisation approach to fractal dimensions of infinite cluster, backbone and cutting bonds for percolation

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Abstract. A Monte Carlo renormalisation group method is presented to study the cluster structure of an infinite cluster at the percolation threshold. Applying this technique to bond percolation on the square lattice, fractal dimensions of an infinite cluster, its backbone and cutting bonds are calculated. This method converges very rapidly with an increase in the size of the cell. We estimate that the fractal dimension of an infinite cluster $D = 1.88 \pm 0.02$, the fractal dimension of its backbone $D_b = 1.62 \pm 0.02$ and the fractal dimension of its cutting bonds $D_c = 0.75 \pm 0.01$, up to the scale factor $b = 10$, in excellent agreement with the large-cell Monte Carlo simulation result.

The percolating infinite cluster is one of the most intensively studied random fractals (Kirkpatrick 1979, Stauffer 1979, 1985, Mandelbrot 1982, Deutscher *et al* 1983, Stanley and Coniglio 1983, Kapitulnik and Deutscher 1984, Stanley and Ostrowsky 1986). Much of our understanding of the structure of clusters for percolation has been obtained by computer simulation. Although such a procedure can yield accurate results and many insights into the structure of such clusters, it nevertheless is not an easy task in comparison with the renormalisation group approach. Here we develop a Monte Carlo renormalisation group method applicable to the cluster structure of percolation. One measure of the structure of an infinite cluster for percolation is the manner in which the N, the total number of bonds in the infinite cluster, scales with the linear dimension L of the cluster, $N \sim L^D$ where D is the fractal dimension of the infinite cluster. The infinite cluster is composed of a backbone through which an electrical current flows and with dangling bonds hanging on it. The number N_b of bonds within the backbone also scales with a different fractal dimension D_b: $N_b \sim L^{D_b}$. Furthermore, there are two kinds of bonds in the backbone: (a) cutting (singly connected) bonds, which have the property that if they are cut, the backbone ceases to conduct and (b) multiply connected bonds, which have the property that they can be cut without interrupting the flow (Stanley 1977). The number N_c of cutting bonds in the infinite cluster scales with $N_c \sim L^{D_c}$. Coniglio (1981, 1982) provided a rigorous argument that for all d, $D_c = 1/\nu$ where the ν represents the connectedness length exponent.

In this letter, we shall calculate the fractal dimensions D, D_b and D_c by using a Monte Carlo renormalisation group method (Lobb and Karasek 1980, Reynolds *et al* 1980, Burkhardt and van Leeuwen 1982). We restrict ourselves to the bond percolation problem on the square lattice.

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The probability that a cell of size b is connected is given by

$$R(b; p) = \sum_k P_k f_k \tag{1}$$

where the summation is over all configurations of the system and P_k is the probability of a particular configuration, while f_k is unity if that configuration spans and zero otherwise. We find that the geometric textures formed by the repeated position-space renormalisation group transformations correspond to the 'regular random fractals' proposed by Martin and Keefer (1985). The geometric texture of an infinite cluster is formed by the repeated transformations with each spanning configuration. By applying N renormalisations to a spanning cluster in the cell of size b^N, the number $N_i(b^N)$ of bonds within the infinite cluster is obtained for a sufficiently large N:

$$\begin{aligned} N_i(b^N) &= n_i(1)n_i(2)\ldots n_i(N) \\ &\sim \langle\langle n_i \rangle\rangle^N \\ &\sim (b^N)^D \end{aligned} \tag{2}$$

where the $n_i(N)$ indicates the number of bonds within a spanning cluster at the Nth renormalisation. Here the $\langle\langle n_i \rangle\rangle$ is the average number of bonds within a spanning cluster:

$$\langle\langle n_i \rangle\rangle \equiv \sum_k n_{i,k} P_k f_k \left(\sum P_k f_k \right)^{-1} \tag{3}$$

where the $n_{i,k}$ represents the number of bonds within the spanning cluster of a particular configuration k.

One can consider a transformation in which one passes from a system of cells of size b_1 to a system of cells of size b_2. Such a 'cell-to-cell' transformation enables one to have a rescaling length b_1/b_2. At the fixed point $R(b_1; p^*) = R(b_2; p^*)$, an incipient infinite cluster appears. The fractal dimension D of the infinite cluster is given by

$$D = \ln(\langle\langle n_i \rangle\rangle_1^* / \langle\langle n_i \rangle\rangle_2^*) / \ln(b_1/b_2) \tag{4}$$

where the asterisk indicates the value at the fixed point. Similarly, we derive the fractal dimensions of the backbone and its cutting bonds:

$$D_b = \ln(\langle\langle n_b \rangle\rangle_1^* / \langle\langle n_b \rangle\rangle_2^*) / \ln(b_1/b_2) \tag{5}$$

$$D_c = \ln(\langle\langle n_c \rangle\rangle_1^* / \langle\langle n_c \rangle\rangle_2^*) / \ln(b_1/b_2) \tag{6}$$

where the n_b and the n_c represent respectively the number of bonds through which an electrical current flows in the spanning cluster and the number of bonds such that if one is cut the entrances are no longer connected to the exits in the cell.

A cell of the type used here is shown in figure 1(a). It can be shown by duality that $p^* = \frac{1}{2}$ independently of the scale factor b (Bernasconi 1978). Normalising the sum in (3) to extend over M connected Monte Carlo realisations and evaluating at $p = p^* = \frac{1}{2}$, we obtain

$$\langle\langle n_i \rangle\rangle^* \simeq (1/M) \sum_{k=1}^{M} n_{i,k} \tag{7}$$

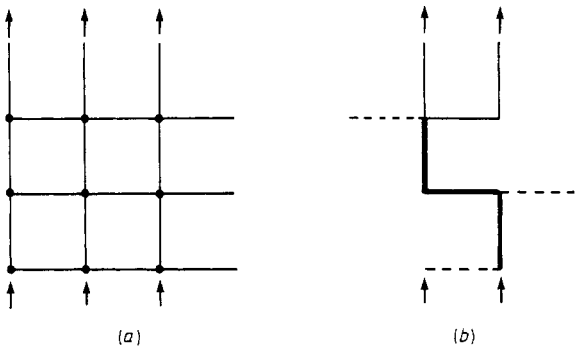

Figure 1. (*a*) Cell used for $b = 3$ transformation. The cell is connected if a continuous path exists from bottom to top. (*b*) Structure of a spanning cluster. Broken, full and bold full lines indicate dangling, multiply connected and cutting bonds.

where the $n_{i,k}$ indicates the number of bonds within the spanning cluster of a connected Monte Carlo realisation. Lattices were generated and studied using the cluster multi-labelling technique of Hoshen and Kopelman (1976) (Stauffer 1985). A connected Monte Carlo realisation is shown in figure 1(*b*). Its backbone, cutting bonds and dangling bonds are respectively indicated by full, bold full and broken lines. Table 1 shows estimates of the fractal dimensions D, D_b and D_c, for the number (10 000) of realisations. The derivations of these estimates are $\sim\Delta D = 0{\cdot}03$, $\Delta D_b = 0{\cdot}04$ and $\Delta D_c = 0{\cdot}06$. These data demonstrate the quick convergence of the method. Within the errors given, the values obtained are indistinguishable from one another. This implies that equation (2) is valid even for small b in the lattices shown by figure 1.

We can improve on the cell-to-cell approach by fitting all of the data to equation (7). As can be seen from the plots of $\ln(\langle\langle n_i\rangle\rangle^*)$, $\ln(\langle\langle n_b\rangle\rangle^*)$ and $\ln(\langle\langle n_c\rangle\rangle^*)$ against $\ln(b)$ (figure 2), the deviation from the expected straight line is small, even for small b. The fits yield

$$D = 1.88 \pm 0.02 \qquad D_b = 1.62 \pm 0.02 \qquad D_c = 0.75 \pm 0.01. \tag{8}$$

This value of the fractal dimension D of an infinite cluster is in good agreement with the earlier results from the large-cell Monte Carlo renormalisation group method with use of a ghost field (Reynolds *et al* 1980) and the large-cell Monte Carlo simulation (Herrmann and Stanley 1984). The value of the fractal dimension D_b of its backbone also agrees with the large-cell Monte Carlo simulation data. The value of the fractal dimension D_c of cutting bonds is in good agreement with the inverse of the connectedness length exponent ν.

Table 1. Estimates of D, D_b and D_c obtained by changing the lattice length scale by a factor b_1/b_2. The number of Monte Carlo realisations considered for each case is 10 000. The statistical uncertainties in each estimate are $\Delta D = 0.03$, $\Delta D_b = 0.04$ and $\Delta D_c = 0.06$.

b_1/b_2	$\frac{4}{3}$	$\frac{5}{4}$	$\frac{6}{5}$	$\frac{7}{6}$	$\frac{8}{7}$	$\frac{9}{8}$	$\frac{10}{9}$
D	1.83	1.88	1.88	1.88	1.91	1.86	1.92
D_b	1.68	1.58	1.65	1.60	1.58	1.64	1.60
D_c	0.76	0.76	0.75	0.75	0.70	0.77	0.72

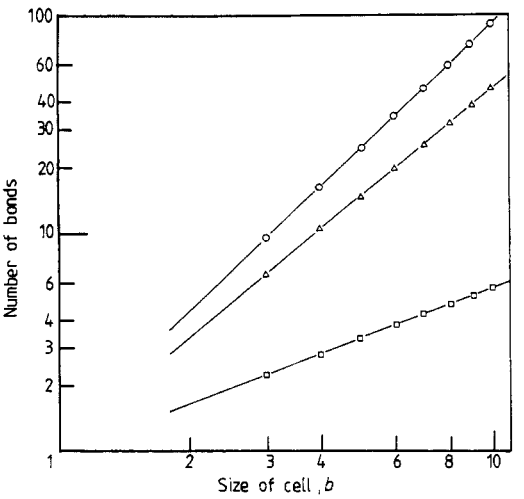

Figure 2. Plots of $\langle\langle n_l \rangle\rangle^*$ (○), $\langle\langle n_b \rangle\rangle^*$ (△) and $\langle\langle n_c \rangle\rangle^*$ (□) against b. The slopes of these lines give estimates for the three fractal dimensions.

In summary, we have used the Monte Carlo renormalisation method to derive the three fractal dimensions (of an infinite cluster, its backbone and cutting bonds) in relation to the cluster structure of an infinite cluster at the percolation threshold. This method gives quick convergence with an increase in the cell size. With relatively small cells, we obtain values in excellent agreement with the large-cell Monte Carlo simulation result.

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